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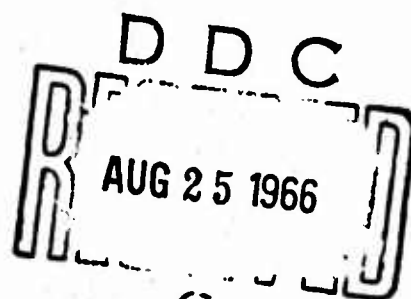
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EDGE COLORINGS IN BIPARTITE GRAPHS

Jon Folkman and D. R. Fulkerson

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PREFACE

This Memorandum deals with a combinatorial problem concerning edge colorings in bipartite graphs. The main results can also be interpreted in terms of sum decompositions of $(0,1)$ -matrices or in terms of multicommodity flows in certain kinds of directed networks.

SUMMARY

An n -list of nonnegative integers $P = (p_1, p_2, \dots, p_n)$ is said to be color-feasible for graph G if the edges of G can be colored (edges on the same vertex having distinct colors) in such a way that exactly p_i edges of G have color i , $i = 1, 2, \dots, n$. The problem studied in this paper is that of determining conditions for color-feasibility of a list P in a bipartite graph G . Necessary and sufficient conditions are obtained in case the n -list P contains at most two distinct positive integers. It is shown that these conditions (while necessary in the general case) are not sufficient if P contains three or more distinct positive integers. For the case of two distinct integers in P , the method of proof leads to an efficient edge-coloring algorithm.

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EDGE COLORINGS IN BIPARTITE GRAPHS

1. INTRODUCTION

The following edge-coloring problem can be posed for any finite graph G . Given a finite sequence of positive integers p_1, p_2, \dots, p_ℓ , when can the edges of G be colored (edges on the same vertex having distinct colors) with ℓ colors in such a way that precisely p_i edges of G have color i , $i = 1, 2, \dots, \ell$? In this general form, the problem is no doubt extremely difficult. Even for the case of bipartite graphs, where one might reasonably expect major simplification to occur, very little seems to be known. The question in this case, rephrased in terms of $(0,1)$ -matrices, becomes: When can a $(0,1)$ -matrix A be written as a sum

$$(1.1) \quad A = P_1 + P_2 + \dots + P_\ell,$$

where each P_i is a permutation matrix of size p_i , that is, P_i has at most one 1 in each row and column and contains p_i 1's? Two special cases of this problem have been examined in [1, 4]. In [1] it is shown that if A has τ 1's and maximum row or column sum k , if p is an integer in the interval $1 \leq p \leq \lceil \frac{\tau}{k} \rceil$, and if r and q are the unique integers such that $r \geq 0$, $0 \leq q < p$, $\tau = (k + r)p + q$, then A can be written as a sum of $k + r$ permutation matrices of size p and one permutation matrix of size q . The main result of

[4] is a combinatorial duality formula for the maximum number of permutation matrices of size m contained in an m -by- n $(0,1)$ -matrix A , where $m \leq n$. Denoting this maximum number by $h(A)$, it is shown in [4] that

$$(1.2) \quad h(A) = \min_{B \subseteq A} \left[\frac{\tau(B)}{s(B)} \right].$$

Here B is an e -by- f submatrix of A , $\tau(B)$ denotes the number of 1's contained in B , $s(B) = e + f - n$, and the minimum is taken over all B such that $s(B) > 0$. The content of formula (1.2) can be rephrased as follows. Let

$$(1.3) \quad p_1 = p_2 = \dots = p_h = m, p_{h+1} = p_{h+2} = \dots = p_l = 1.$$

Then a necessary and sufficient condition for a decomposition (1.1) is that for any e -by- f submatrix B of A , $e = 0, 1, \dots, m$, $f = 0, 1, \dots, n$, we have

$$(1.4) \quad \tau(B) \geq \sum_{j=(m-e)+(n-f)+1}^{\infty} p_j^*.$$

In (1.4) p_j^* denotes the number of integers p_i that are greater than or equal to j , that is, the p -sequence and the p^* -sequence are conjugate partitions of the integer $p_1 + p_2 + \dots + p_l$. It is also understood that equality holds in (1.4) for $B = A$.

Although conditions (1.4) are necessary for a decomposition (1.1) in which P_i contains p_i 1's, it can be seen from examples that they are not in general sufficient. We

shall prove in Secs. 2 and 3, however, that these conditions are sufficient if

$$(1.5) \quad p_1 = p_2 = \dots = p_h = p, \quad p_{h+1} = p_{h+2} = \dots = p_l = q,$$

thereby generalizing the main result of [4]. Our attempts to find necessary and sufficient conditions for arbitrary p_i have not been successful. What little information we have on the general case is presented in Sec. 4.

There is a kind of coloring problem involving matroids for which conditions analogous to (1.4) are known to be both necessary and sufficient. A matroid $M = (E, F)$ is a finite set E of elements and a family F of subsets of E , called independent sets, such that (1) every subset of an independent set is independent, and (2) for every set $X \subset E$, all maximal independent subsets of X have the same cardinality, called the rank $r(X)$ of X . It is known [2] that the elements E of matroid M can be partitioned into independent sets I_1, I_2, \dots, I_l of respective sizes p_1, p_2, \dots, p_l if and only if, for every $X \subset E$,

$$(1.6) \quad |X| \geq \sum_{j=r(\bar{X})+1}^{\infty} p_j^*.$$

Here $\bar{X} = E - X$, $|X|$ denotes the cardinality of set X , and equality is assumed to hold for $X = E$. Thus the coloring problem is one in which a set of elements can have the same color if they form an independent subset of E , and we are asked to color elements of E in such a way that p_i elements

have color i , $i = 1, 2, \dots, t$. If we say that a set of edges in the bipartite graph G having edge set E (a set of 1's in the $(0,1)$ -matrix A) is "independent" if it forms a matching, that is, if no two edges of the set are on the same vertex (no two 1's lie in the same row or column), the family F of "independent" sets thus defined satisfies axiom (1) for matroids, but does not satisfy axiom (2). However, an analog of (1.6) for the resulting coloring problem would be

$$(1.7) \quad |X| \geq \sum_{j=\rho(\bar{X})+1}^{\infty} p_j^*, \quad \text{all } X \subset E,$$

where $\rho(\bar{X})$ denotes the maximum size of a matching in set \bar{X} of edges. Conditions (1.7), which appear on the surface to be stronger than (1.4), are again necessary, but not sufficient, for the desired coloring. Indeed, using the König theorem on maximum matchings in bipartite graphs, it can be shown that (1.4) and (1.7) are actually equivalent systems of inequalities. Thus the matroid result mentioned above extends in only a limited way to the non-matroidal situation we are concerned with here.

2. A DECOMPOSITION THEOREM FOR BIPARTITE GRAPHS

Let $G = [M, N; E]$ be a bipartite graph with edge set E and vertex "parts" $M = \{1, 2, \dots, m\}$, $N = \{1, 2, \dots, n\}$. Thus each edge of G joins a vertex of M to a vertex of N . Throughout this and the following section we shall need to single out subsets of E of the following kind—all the edges of G that join vertices in $X \subset M$ to vertices in $Y \subset N$. We denote such a set of edges by (X, Y) . Thus $E = (M, N)$.

Let $G' = [M, N; E']$ and $G'' = [M, N; E'']$ be subgraphs of G and suppose that E', E'' is a partition of E , empty sets not being excluded. We then write $G = G' + G''$, and say that G is the sum of its subgraphs G', G'' . (If we let A, A', A'' be the m -by- n adjacency matrices for G, G', G'' , respectively, we have $A = A' + A''$.) Let the degree in graph G of vertex $i \in M$ be denoted by r_i , that of vertex $j \in N$ by s_j . Similarly, let r'_i, s'_j , and r''_i, s''_j denote degrees in G' and G'' , respectively. The question we raise and answer in this section is the following. For each $i \in M$ and $j \in N$ let ρ'_i, σ'_j and ρ''_i, σ''_j be specified nonnegative integers. We also specify two nonnegative integers τ', τ'' . When can we write $G = G' + G''$, where the degrees in G' satisfy $r'_i \leq \rho'_i, s'_j \leq \sigma'_j$, the degrees in G'' satisfy $r''_i \leq \rho''_i, s''_j \leq \sigma''_j$, and G' has τ' edges, G'' has τ'' edges? As we shall see in the next section, the answer to this question leads to a solution of the edge-coloring problem for bipartite graphs if the integers

p_1, p_2, \dots, p_l satisfy (1.5). Moreover, the method of proof, which uses basic results of network flow theory [3], provides an efficient edge-coloring algorithm in this special case.

Theorem 2.1. Let $G = [M, N; E]$ be a bipartite graph having degrees $r_i, i \in M$, and $s_j, j \in N$, and $\tau = |E|$ edges. Further, let $\rho_i', \sigma_j', \rho_i'', \sigma_j'', \tau', \tau''$ be specified nonnegative integers satisfying

$$(2.1) \quad \tau = \tau' + \tau'';$$

$$(2.2) \quad r_i \leq \rho_i' + \rho_i'', \quad i \in M;$$

$$(2.3) \quad s_j \leq \sigma_j' + \sigma_j'', \quad j \in N.$$

Then

$$(2.4) \quad G = G' + G'',$$

where $G'(G'')$ has $\tau'(\tau'')$ edges and degrees $r_i' \leq \rho_i'$ for $i \in M$ ($r_i'' \leq \rho_i''$ for $i \in M$) and $s_j' \leq \sigma_j'$ for $j \in N$ ($s_j'' \leq \sigma_j''$ for $j \in N$) if and only if, for each $X \subset M, Y \subset N$, we have

$$(2.5) \quad \tau' - \sum_{i \in \bar{X}} \rho_i' - \sum_{j \in \bar{Y}} \sigma_j' \leq |(X, Y)|,$$

$$(2.6) \quad \tau'' - \sum_{i \in \bar{X}} \rho_i'' - \sum_{j \in \bar{Y}} \sigma_j'' \leq |(X, Y)|,$$

$$(2.7) \quad |(X, Y)| \leq \sum_{i \in X} \rho_i' + \sum_{j \in Y} \sigma_j'',$$

$$(2.8) \quad |(X, Y)| \leq \sum_{i \in X} \rho_i'' + \sum_{j \in Y} \sigma_j'.$$

Here $\bar{X} = M - X$, $\bar{Y} = N - Y$, and $|(X, Y)|$ denotes the number of edges joining X to Y in G.

Proof. The necessity of each of the conditions (2.5)–(2.8) can easily be verified directly. Suppose (2.4) holds with G' and G'' as specified, and let t' denote the number of edges joining X to Y in G' , t'' the number of edges joining X to Y in G'' . Then

$$r' \leq \sum_{i \in \bar{X}} r_i' + \sum_{j \in \bar{Y}} s_j' + t' \leq \sum_{i \in \bar{X}} \rho_i' + \sum_{j \in \bar{Y}} \sigma_j' + |(X, Y)|,$$

$$|(X, Y)| = t' + t'' \leq \sum_{i \in X} r_i' + \sum_{j \in Y} s_j'' \leq \sum_{i \in X} \rho_i' + \sum_{j \in Y} \sigma_j'',$$

verifying (2.5) and (2.7). Similarly for (2.6) and (2.8).

The sufficiency of conditions (2.5)–(2.8) can be established using known results about flows in networks. We begin by imbedding the graph G in an appropriate flow network G^* as follows. The vertex set V of G^* consists of x_1, x_2, \dots, x_m , corresponding to part M of G , and y_1, y_2, \dots, y_n , corresponding to part N of G , plus two additional vertices that we label a and b . The directed

edges of G^* are those ordered pairs (x_i, y_j) that correspond to the edges of G , plus the ordered pairs (a, x_i) for $i \in M$, (y_j, b) for $j \in N$, and (b, a) . We now impose nonnegative integral lower bounds $l(x, y)$ and capacities (upper bounds) $c(x, y)$ on the amount of flow $f(x, y)$ in edge (x, y) of G^* as follows:

$$\begin{aligned}
 (2.9) \quad l(a, x_i) &= \max(0, r_i - \rho_i''), & c(a, x_i) &= \rho_i', \\
 l(x_i, y_j) &= 0, & c(x_i, y_j) &= 1, \\
 l(y_j, b) &= \max(0, s_j - \sigma_j''), & c(y_j, b) &= \sigma_j', \\
 l(b, a) &= \tau', & c(b, a) &= \tau'.
 \end{aligned}$$

Then an integral-valued flow f satisfying Kirchoff's conservation law at all vertices of G^* and also satisfying the prescribed bounds $l(x, y) \leq f(x, y) \leq c(x, y)$ for all edges (x, y) of G^* picks out subgraphs G' , G'' of G satisfying the requirements of the theorem by putting edge (i, j) of G in G' or G'' according as $f(x_i, y_j) = 1$ or $f(x_i, y_j) = 0$. It therefore suffices to show that (2.5)-(2.8) imply the existence of such a flow.

It is known [5] that such a flow exists if, for every subset $Z \subset V$ of the vertices of G^* , the sum of the lower bounds on edges from Z to $\bar{Z} = V - Z$ is less than or equal to the sum of the capacities on edges from \bar{Z} to Z :

$$(2.10) \quad \sum_{\substack{x \in Z \\ y \in \bar{Z}}} r(x, y) \leq \sum_{\substack{x \in \bar{Z} \\ y \in Z}} c(x, y).$$

We shall show that (2.5)–(2.8) imply (2.10).

Let $Z \subset V$. First assume $a \in Z$, $b \in \bar{Z}$. Let \bar{S} denote the subset of indices $i \in M$ such that $x_i \in \bar{Z}$, and T the subset of indices $j \in N$ such that $y_j \in Z$. Then (2.10) may be written as

$$(2.11) \quad \sum_{i \in \bar{S}} \max(0, r_i - \rho_i'') + \sum_{j \in T} \max(0, s_j - \sigma_j'') \leq \tau' + |(\bar{S}, T)|.$$

Let $\bar{X} \subset \bar{S}$ be that subset of \bar{S} on which $r_i - \rho_i'' > 0$, and let $\bar{Y} \subset T$ be that subset of T on which $s_j - \sigma_j'' > 0$. Then (2.11) becomes

$$(2.12) \quad \tau + |(\bar{S}, T)| - \sum_{i \in \bar{X}} r_i - \sum_{j \in \bar{Y}} s_j \geq \tau'' - \sum_{i \in \bar{X}} \rho_i'' - \sum_{j \in \bar{Y}} \sigma_j''.$$

But the left-hand side of (2.12) is at least $|(X, Y)|$, where $X = M - \bar{X}$, $Y = N - \bar{Y}$, and hence (2.6) implies (2.12).

Next suppose $a \in Z$, $b \in Z$. Let \bar{S} denote the subset of indices $i \in M$ such that $x_i \in \bar{Z}$, let \bar{Y} denote the subset of indices $j \in N$ such that $y_j \in Z$, and let $Y = N - \bar{Y}$. Then (2.10) may be written as

$$(2.13) \quad \sum_{i \in \bar{S}} \max(0, r_i - \rho_i'') \leq |(\bar{S}, \bar{Y})| + \sum_{j \in Y} \sigma_j'.$$

Let $X \subset \bar{S}$ be that subset of \bar{S} on which $r_i - \rho_i'' > 0$, and $\bar{X} = M - X$. Then (2.13) becomes

$$(2.14) \quad \sum_{i \in X} r_i - |(\bar{S}, \bar{Y})| \leq \sum_{i \in X} \rho_i'' + \sum_{j \in Y} \sigma_j'.$$

Since the left-hand side of (2.14) is at most $|(X, Y)|$, we see that (2.8) implies (2.14).

The remaining two cases, $a \in \bar{Z}$, $b \in Z$ and $a \in \bar{Z}$, $b \in \bar{Z}$, can be dealt with similarly. In the first case, (2.5) implies (2.10); in the second, (2.7) implies (2.10).

This completes the proof of Theorem 2.1.

In Corollaries 2.2 and 2.3 below, we specialize the primed parameters occurring in (2.1)–(2.3) in order to note simplifications that occur in the existence conditions (2.5)–(2.8). We shall be particularly interested in the case of constant bounds on degrees in G' and in G'' .

Corollary 2.2. Let $\rho_i' = k'$ and $\rho_i'' = k''$, all $i \in M$, and let $\sigma_j' = k'$, $\sigma_j'' = k''$, all $j \in N$. Then there is a decomposition (2.4) if and only if the inequalities

$$(2.5a) \quad \tau' - k'(|\bar{X}| + |\bar{Y}|) \leq |(X, Y)|,$$

$$(2.6a) \quad \tau'' - k''(|\bar{X}| + |\bar{Y}|) \leq |(X, Y)|$$

hold for all $X \subset M$, $Y \subset N$.

Proof. It suffices to show that (2.7) and (2.8) hold automatically. Suppose that (2.7) fails for some $X \subset M$, $Y \subset N$:

$$(2.15) \quad |(X, Y)| > k'|X| + k''|Y|.$$

Let $k = \max(r_1, \dots, r_m, s_1, \dots, s_n)$ be the maximum degree in G . Thus $k \leq k' + k''$ by (2.2) and (2.3). Then

$$(2.16) \quad -|(X, Y)| \geq -k|X|,$$

$$(2.17) \quad -|(X, Y)| \geq -k|Y|.$$

Adding (2.15) and (2.16), and (2.15) and (2.17), yields

$$0 > (k' - k)|X| + k''|Y| \geq k''(|Y| - |X|),$$

$$0 > k'|X| + (k'' - k)|Y| \geq k'(|X| - |Y|),$$

a contradiction. Hence (2.7) holds. Similarly for (2.8).

If we further specialize parameters by taking $k' = 1$, $k'' = k - 1$, where k is the maximum degree in G , a generalization of a result due to Dulmage and Mendelsohn [1] is obtained. In this case there always exists a value of τ' that produces a decomposition (2.4). Indeed, it is shown in [1] that if the bipartite graph G has τ edges and maximum degree k , there exists a matching in G of size $\tau' = \lfloor \tau/k \rfloor$ that "hits" all vertices of degree k ; that is, each vertex of degree k is incident with some edge of the matching. Corollary 2.3 below describes the full range of values of τ' for which such a matching exists.

Corollary 2.3. Let $k > 0$ be the maximum degree in the

bipartite graph $G = [M, N; E]$ having $\tau = |E|$ edges, and let

$$(2.18) \quad \max_{\substack{X \subset M \\ Y \subset N}} \{ \tau - |(X, Y)| - (k-1)(|\bar{X}| + |\bar{Y}|) \} = \sigma,$$

$$(2.19) \quad \min_{\substack{X \subset M \\ Y \subset N}} \{ |(X, Y)| + |\bar{X}| + |\bar{Y}| \} = \rho.$$

Then, for each τ' in the interval

$$(2.20) \quad \sigma \leq \tau' \leq \rho,$$

there exists a decomposition $G = G' + G''$ where G' is a
matching of size τ' and G'' has maximum degree $k-1$. In
particular, the integer $\tau' = \lfloor \tau/k \rfloor$ satisfies (2.20).

Proof. We show first that the interval (2.20) is nonempty. Let $\tau' = \lfloor \tau/k \rfloor$. If $\tau' > \rho$, then there are $X \subset M, Y \subset N$ such that

$$\tau > k\{|(X, Y)| + |\bar{X}| + |\bar{Y}|\},$$

a contradiction. Hence $\tau' \leq \rho$. Next suppose $\tau' < \sigma$. Then there are $X \subset M, Y \subset N$ such that

$$\tau - \lfloor \tau/k \rfloor > |(X, Y)| + (k-1)(|\bar{X}| + |\bar{Y}|)$$

and hence

$$\tau(k-1) > k|(X, Y)| + k(k-1)(|\bar{X}| + |\bar{Y}|),$$

again a contradiction. Hence $\sigma \leq [\tau/k] \leq \rho$. Corollary 2.3 now follows from Corollary 2.2.

The nonnegative integers σ and ρ defined in (2.18) and (2.19) can be described in other ways. It is easy to see that the minimum in (2.19) occurs for X, Y such that (X, Y) is empty, and hence ρ is the minimum number of vertices that cover all edges in G . By the König theorem, this is equal to the size of a maximum matching in G (frequently called the term rank of G). The integer σ can also be described in terms of certain matchings in G . If we let $S \subset M$, $T \subset N$ be the vertices of maximum degree k in G , and let $\rho(X, Y)$ denote the size of a maximum matching in the subgraph of G having edge set (X, Y) , then it can be shown that

$$(2.21) \quad \sigma = \rho(S, N) + \rho(M, T) - \rho(S, T).$$

We conclude this section with some further discussion of the existence conditions (2.5)–(2.8) of Theorem 2.1. Our first comment concerns (2.7), (2.8). We noted in Corollary 2.2 that these conditions could be dispensed with in the case of constant bounds on degrees in G' and G'' . In the general case, however, it can be seen from examples that these conditions are essential. Our second comment concerns interpretations of conditions (2.5)–(2.8), viewed individually. Suppose we know, for example, that (2.5) holds. What does this say about G , if anything? It is not difficult to see (by taking $\rho_i'' = \sigma_j'' = \infty$ in Theorem 2.1,

for instance) that inequalities (2.5) are equivalent to the existence of a decomposition $G = G' + G''$ where $r_i' \leq \rho_i'$ for $i \in M$, $s_j' \leq \sigma_j'$ for $j \in N$, and G' has τ' edges. Similarly for (2.6). Conditions (2.7), on the other hand, are equivalent to the existence of a decomposition $G = G' + G''$, where $r_i' \leq \rho_i'$ for $i \in M$ and $s_j'' \leq \sigma_j''$ for $j \in N$, and similarly for (2.8). For example, consider (2.7). Let ρ_i' and σ_j'' be specified nonnegative integers, and take $\rho_i'' = \sigma_j' = \infty$. Then (2.8) clearly holds. Define

$$(2.22) \quad \tau' = \min_{X \subset M} \{ |(X, N)| + \sum_{i \in \bar{X}} \rho_i' \} = |(X_0, N)| + \sum_{i \in \bar{X}_0} \rho_i',$$

so that (2.5) is valid. Suppose that (2.6) were violated for some $X \subset M$, $Y \subset N$. Then clearly $X = M$, and hence

$$\tau' < \tau - |(M, Y)| - \sum_{j \in \bar{Y}} \sigma_j'' = |(M, \bar{Y})| - \sum_{j \in \bar{Y}} \sigma_j''.$$

By (2.22) we have

$$\begin{aligned} |(X_0, N)| + \sum_{i \in \bar{X}_0} \rho_i' &< |(M, \bar{Y})| - \sum_{j \in \bar{Y}} \sigma_j'' \\ \sum_{i \in \bar{X}_0} \rho_i' + \sum_{j \in \bar{Y}} \sigma_j'' &< |(M, \bar{Y})| - |(X_0, N)| \leq |(X_0, \bar{Y})|, \end{aligned}$$

contradicting (2.7). Hence (2.7) implies $G = G' + G''$, where $r_i' \leq \rho_i'$, $s_j'' \leq \sigma_j''$, and G' has the number of edges given by (2.22). To sum up, Theorem 2.1 can be viewed as saying that

if G can be decomposed in four different ways, each of which satisfies a certain subset of requirements in the theorem, there will be a single decomposition of G satisfying all requirements in the theorem.

We state this result for the situation of Corollary 2.2 explicitly.

Corollary 2.4. Let the bipartite graph G have τ edges and maximum degree k . Suppose $\tau = \tau' + \tau''$ for nonnegative integers τ', τ'' , and let k', k'' be nonnegative integers satisfying $k \leq k' + k''$. If G has a subgraph H' having τ' edges and degrees not exceeding k' , and also a subgraph H'' having τ'' edges and degrees not exceeding k'' , then $G = G' + G''$ where $G'(G'')$ has $\tau'(\tau'')$ edges and degrees not exceeding $k'(k'')$.

3. EDGE COLORINGS

In this section we assume that (1.5) holds and apply the results of Sec. 2 to the edge-coloring problem described in Sec. 1.

We say that a sequence of positive integers p_1, p_2, \dots, p_ℓ is color-feasible in graph G if there is an edge coloring of G in which precisely p_i edges have color i , $i = 1, 2, \dots, \ell$.

Theorem 3.1. Assume

$$(3.1) \quad p_1 = \dots = p_h = p, \quad p_{h+1} = \dots = p_\ell = q.$$

The sequence (3.1) is color-feasible in the bipartite graph $G = [M, N; E]$ if and only if the inequalities

$$(3.2) \quad |(X, Y)| \geq \sum_{j=|\bar{X}|+|\bar{Y}|+1}^{\infty} p_j^*$$

hold for all $X \subset M, Y \subset N$. Here the p^* -sequence is the conjugate of (3.1), and equality is assumed to hold in (3.2) for $X = M, Y = N$.

Proof. Suppose the sequence p_1, p_2, \dots, p_ℓ is color-feasible in G . The number of edges in $E - (X, Y)$ having color i is at most $|\bar{X}| + |\bar{Y}|$; hence the number of edges in (X, Y) having color i is at least $\max(0, p_i - |\bar{X}| - |\bar{Y}|)$. Summing over i yields

$$|(X, Y)| \geq \sum_{i=1}^{\ell} \max(0, p_i - |\bar{X}| - |\bar{Y}|) = \sum_{j=|\bar{X}|+|\bar{Y}|+1}^{\infty} p_j^*.$$

Now assume (3.1) and (3.2). We shall first apply Corollary 2.2 with

$$h = k', \quad \iota - h = k'', \quad ph = \tau', \quad q(\iota - h) = \tau''.$$

To this end, we note that by (3.2), with $X = M$, $Y = N$, we have $\tau' + \tau'' = |(M, N)| = \tau$. Also, taking $X = M - \{i\}$, $Y = N$ in (3.2) yields

$$|(M - \{i\}, N)| \geq \sum_{j=2}^{\infty} p_j^*,$$

and hence

$$r_i = |(i, N)| \leq p_1^* = \iota, \quad i \in M.$$

Similarly, $s_j \leq \iota$ for $j \in N$. Thus the maximum degree k in G satisfies

$$k \leq k' + k'' = \iota.$$

We now check that (3.2) implies the existence conditions (2.5a) and (2.6a) of Corollary 2.2. Let X, Y be arbitrary subsets of M, N , respectively. Then

$$(3.3) \quad |(X, Y)| \geq \sum_{j=|X|+|Y|+1}^{\infty} p_j^* \geq ph - h(|\bar{X}| + |\bar{Y}|),$$

verifying (2.5a). Similarly,

$$(3.4) \quad |(X, Y)| \geq \sum_{j=|X|+|Y|+1}^{\infty} p_j^* \geq q(\iota - h) - (\iota - h)(|\bar{X}| + |\bar{Y}|).$$

It follows from Corollary 2.2 that $G = G' + G''$ where G' has ph edges and degrees not exceeding h , and G'' has $q(\iota - h)$

edges and degrees not exceeding $l - h$.

We can further decompose G' , and also G'' , by use of the following lemma.

Lemma 3.2. Let G be a bipartite graph having τ edges and maximum degree k . Then G decomposes into a sum of h matchings, each of size p , if and only if $\tau = ph$ and $k \leq h$.

The necessity of these conditions is obvious. The sufficiency is a consequence of the Dulmage-Mendelsohn theorem described in Sec. 1. Sufficiency can also be established by induction on h , using Corollary 2.3, as follows. The case $h = 1$ is trivial. Assume the validity of the lemma for $h - 1$ and consider h . If $k < h$, then $\lceil \tau/k \rceil \geq \lceil \tau/h \rceil = p$. By Corollary 2.3, G has a matching G_1 of size p . Moreover, the graph $G - G_1$ obtained by deleting edges of G_1 from G has $p(h - 1)$ edges and the assumption $k < h$ implies that $G - G_1$ has maximum degree less than or equal to $h - 1$. On the other hand, if $k = h$, Corollary 2.3 implies that G has a matching G_1 of size p that hits all vertices of degree h . Thus again $G - G_1$ has $p(h - 1)$ edges and maximum degree $h - 1$. The lemma now follows from the induction assumption.

We return to the decomposition $G = G' + G''$ reached prior to the statement of Lemma 3.2 in the proof of Theorem 3.1. It follows from the lemma that G' decomposes into a sum of h matchings, each of size p , and that G''

decomposes into a sum of $\ell - h$ matchings, each of size q . This completes the proof of Theorem 3.1.

By taking $p_{h+1} = \dots = p_\ell = 1$ in (3.1), we obtain the following corollary.

Corollary 3.3. Let $h(G, p)$ denote the maximum number of disjoint matchings, each of size p , contained in the bipartite graph $G = [M, N; E]$. Then

$$(3.5) \quad h(G, p) = \min_{\substack{X \subseteq M \\ Y \subseteq N}} \left[\frac{|(X, Y)|}{p - |\bar{X}| - |\bar{Y}|} \right],$$

the minimum in (3.5) being taken over all X, Y such that $p - |\bar{X}| - |\bar{Y}| > 0$.

This result is a direct generalization of formula (1.2) for the case $p = |M| = m \leq n$.

4. SOME REMARKS ON THE GENERAL CASE

It can be seen from examples that conditions (3.2) are not sufficient for the sequence p_1, p_2, \dots, p_l to be color-feasible in a bipartite graph if this sequence contains three or more distinct positive integers. For instance, consider the tree in Fig. 4.1 having the adjacency matrix shown there. This graph satisfies (3.2) for the

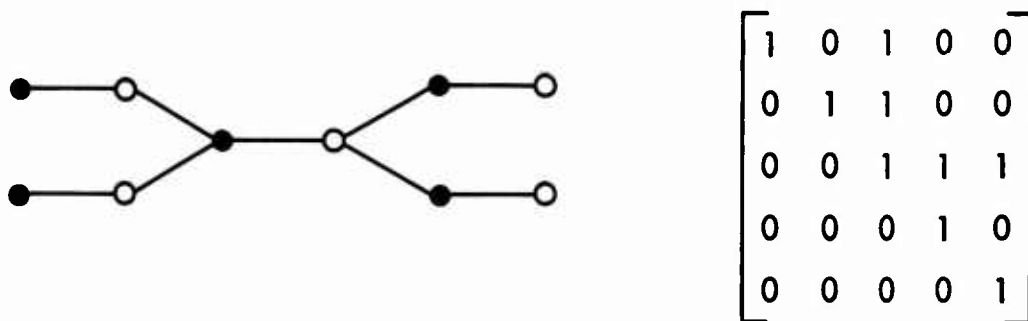


Fig.4.1

sequence 5, 3, 1, but this sequence is not color-feasible. Each of the sequences 5, 2, 2 and 4, 4, 1 satisfies (3.2) and is color-feasible.

Let $P = (p_1, p_2, \dots, p_n)$ and $Q = (q_1, q_2, \dots, q_n)$ be two sequences of nonnegative integers. The sequence P is said to majorize Q , written $P \succ Q$, provided that with subscripts renumbered in accordance with

$$(4.1) \quad p_1 \geq p_2 \geq \dots \geq p_n, \quad q_1 \geq q_2 \geq \dots \geq q_n,$$

we have

$$(4.2) \quad \sum_{i=1}^e p_i \geq \sum_{i=1}^e q_i, \quad e = 1, 2, \dots, n-1,$$

$$(4.3) \quad \sum_{i=1}^n p_i = \sum_{i=1}^n q_i.$$

If $P \succ Q$ and $Q \succ P$, we say that P and Q are equivalent and write $P \approx Q$. Thus $P \approx Q$ means that with the numbering selected in (4.1), we have $p_i = q_i$, $i = 1, 2, \dots, n$. The majorization relation can be viewed as a partial order on the set of all n -lists (unordered n -tuples) of nonnegative integers satisfying (4.3).

It follows from Theorem 3.1 that if P and Q each contains at most two distinct positive integers, and if P is color-feasible and $P \succ Q$, then Q is also color-feasible. This is in fact generally so, without any special assumptions on P and Q . To prove this, we first make a definition and establish a lemma. Let $P = (p_1, p_2, \dots, p_n)$, and suppose $p_i > p_j$. Then the sequence P' obtained from P by defining

$$(4.4) \quad \begin{aligned} p'_i &= p_i - 1, \\ p'_j &= p_j + 1, \\ p'_k &= p_k, \quad k \neq i, j, \end{aligned}$$

satisfies $P \succ P'$. Moreover, if $p_i \geq p_j + 2$, we have $P \not\approx P'$. We call the transformation (4.4) a transfer from i to j on P .

Lemma 4.1. If $P \succ Q$, then P can be transformed into Q by a finite sequence of transfers.

Proof. Select the numbering so that P and Q are monotone decreasing, and suppose $P \not\succeq Q$. Let ℓ be the last integer in the interval $1 \leq \ell \leq n - 1$ for which strict inequality holds in (4.2). Then $p_{\ell+1} < q_{\ell+1}$. There are integers in the interval $1 \leq i \leq \ell$ for which $p_i > q_i$. Let k be the last such. Thus $p_k > q_k \geq q_{\ell+1} > p_{\ell+1}$. Let P' be obtained from P by a transfer from k to $\ell + 1$. Then $P \succ P' \succ Q$ and $P \not\succeq P'$. Repetition of this process establishes the lemma.

Using Lemma 4.1, it is easy to prove Theorem 4.2, below. In Theorem 4.2 it is unnecessary to suppose the graph G to be bipartite.

Theorem 4.2. Let G be an arbitrary graph and suppose that $P = (p_1, p_2, \dots, p_n)$ is color-feasible in G . If $P \succ Q = (q_1, q_2, \dots, q_n)$, then Q is also color-feasible in G .

Proof. By Lemma 4.1, it is enough to prove that if P' is obtained from P by a transfer from i to j , then P' is also color-feasible. Let $G = G_1 + G_2 + \dots + G_n$, where G_i is a matching of size p_i , $i = 1, 2, \dots, n$. Consider the matchings G_i, G_j , where $p_i > p_j$. Each connected component of the graph $G_i + G_j$ is either an even circuit or a chain, with edges alternately in G_i and G_j . Moreover, since $p_i > p_j$, at least one component must be an odd chain

having first and last edges in G_i . Let G'_i and G'_j be obtained from G_i and G_j by interchanging G_i -edges and G_j -edges in this chain. This produces a coloring of G in which p'_i edges have color i .

In view of Theorem 4.2, one possible approach to the general edge-coloring problem might be in the direction of attempting to characterize those lists of nonnegative integers that are color-feasible and maximal in the sense of majorization, that is, are not majorized by any other color-feasible list. (In the example of Fig. 4.1, for instance, $(5, 2, 2)$ and $(4, 4, 1)$ are the only two such lists.) However, even for the case of bipartite graphs, we don't know how to construct one such list, let alone all of them. The main information we have in this direction is contained in Theorem 4.3, below, the proof of which is modeled on König's proof that the edges of a bipartite graph having maximum degree k can always be colored with k colors [6].

Theorem 4.3. Let G be a bipartite graph having maximum degree k . Then each maximal color-feasible list for G contains exactly k positive members.

Proof. Let p_1, \dots, p_r be a maximal color-feasible list for G and assume that $p_1 \geq p_2 \geq \dots \geq p_r > 0$. Since some vertex of G has k edges incident to it, we must have $r \geq k$. Suppose that $r > k$. Let $G = G_1 + \dots + G_r$ be a decomposition of G into matchings such that G_i has p_i edges. Let (u, v) be an edge in G_{k+1} . Now u is incident to at

most $k - 1$ edges of G other than (u, v) . Hence, there is an s with $1 \leq s \leq k$ such that u is incident to no edge of G_s . Similarly, there is a t with $1 \leq t \leq k$ such that v is incident to no edge of G_t . Interchanging u and v if necessary, we may assume that $s \leq t$.

We will now construct a decomposition $G = G'_1 + \dots + G'_r$ of G into matchings with the following properties:

$$(4.5) \quad \begin{cases} G'_i = G_i & \text{for } i \neq s, t, \\ \text{number of edges of } G'_s \geq \text{number of edges of } G_s, \\ \text{neither } u \text{ nor } v \text{ is incident to an edge of } G'_t. \end{cases}$$

If u is not incident to any edge of G_t , take $G'_i = G_i$ for all i . If u is incident to some edge of G_t , let U be the component of $G_s + G_t$ containing u . Since u is incident to no edge of G_s , U is a chain with u as an endpoint. Suppose v is in U . Then v would have to be the other endpoint of U . Furthermore, the number of edges in U would have to be even, since the edges are alternately in G_t and G_s , with the first edge in G_t and the last one in G_s . Hence, U together with the edge (u, v) would form a circuit in G of odd length. This is impossible because G is bipartite. Therefore v is not in U . Now let $G'_i = G_i$ for $i \neq s, t$, and let G'_s and G'_t be obtained from G_s and G_t by interchanging G_s -edges and G_t -edges in the chain U . Conditions (4.5) now hold.

Since neither u nor v is incident to an edge of G'_t , we may define a decomposition $G = G''_1 + \dots + G''_r$ of G into

matchings by setting

$$(4.6) \left\{ \begin{array}{l} G''_t = G'_t \cup \{(u, v)\}, \\ G''_{k+1} = G'_{k+1} - \{(u, v)\}, \\ G''_i = G'_i \quad \text{for } i \neq t, k+1. \end{array} \right.$$

Let q_i be the number of edges in G''_i . Then q_1, q_2, \dots, q_r is color-feasible for G . Furthermore, by (4.5) and (4.6) we have

$$\begin{aligned} q_i &= p_i \quad \text{for } i \neq s, t, k+1, \\ q_s &= p_s + e, \\ q_t &= p_t - e + 1, \\ q_{k+1} &= p_{k+1} - 1, \end{aligned}$$

where $e \geq 0$ is the excess of the number of edges in G'_s over the number of edges in G_s . Since $s \leq t < k+1$ it now follows that

$$\begin{aligned} \sum_{i=1}^j q_i &\geq \sum_{i=1}^j p_i \quad \text{for } 1 \leq j \leq r, \\ \sum_{i=1}^k q_i &> \sum_{i=1}^k p_i. \end{aligned}$$

Since the list P is in monotone decreasing order, this implies that $Q \succ P$ and $Q \not\leq P$, contradicting the assumption that P was a maximal color-feasible list. Hence $r = k$.

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10. ABSTRACT <p>The determination of conditions for color-feasibility of a list of nonnegative integers, P, in a bipartite graph, G. Necessary and sufficient conditions are obtained in case the n-list P contains at most two distinct positive integers. It is shown that these conditions (while necessary in the general case) are not sufficient if P contains three or more distinct positive integers. For the case of two distinct integers in P, the method of proof leads to an efficient edge-coloring algorithm. The main results can also be interpreted in terms of sum decompositions of $(0,1)$-matrices or in terms of multicommodity flows in certain kinds of connected networks.</p>		11. KEY WORDS Graph theory Mathematics Algorithms Network theory	